Objectives

• Use an example problem to illustrate various astrodinamic techniques you’ll need to know

• Introduce you to the various topics that the text covers in more detail
Problem Scenario

• Determine when you can see a satellite from a ground site

• What we’ll need to understand
  – Time
  – Coordinate systems
  – Propagation
  – Orbit Determination
  – ... and some others 😊
What we’ll cover

• Fundamental Concepts
  – Time and Coordinate Systems
• Newton
  – Equations of Motion
• Kepler
  – Equation
  – Problem
  – Satellite state
• Perturbations/Propagation
  – Special
  – General
• Orbit Determination and Estimation
• Applications
Fundamentals of Astrodynamics and Applications Third Edition

Space Technology Library (Vol 21), Microcosm Press/Springer

By David A. Vallado
Center for Space Standards and innovation


Published Spring 2007


US$ 60.00
JOHANNES KEPLER'S UPHILL BATTLE

...SO, YOU SEE, THE ORBIT OF A PLANET IS ELLIPTICAL.

WHAT'S AN ORBIT?
WHAT'S A PLANET?
WHAT'S ELLIPTICAL?
My Objective with the Book

• Cover
  – Fundamentals
  – Some advanced material
• Bridge the gap in between
• Details
  – Consistent notation
Chapter 3

- Fundamental Concepts
- Newton
- Kepler
- Perturbations
- Orbit Determination
- Applications
Time and Coordinate Systems

- Essential, but not terribly exciting
What Time is it?

14:28
- Ok – That specifies that it’s afternoon
- But what time zone?
  - Mountain Time is 6/7 hours before UTC (Greenwich, Zulu)
    - Need to specify
      » Daylight Savings
      » Standard Time
- Is that all? … No!
  - TAI, TT (TDT), TDB, TCB, TCG, GPS, …
Solar and Sidereal Time

- **Earth**
- **Sun**
- **Stars**

**References Directions**

**Sidereal day (23h 56m 4.0905s)**

**Solar day (24h)**
Greenwich and Local Times

Local Meridian

Star

$LHA_{star}$

$\theta_{LST}$

$\lambda$

$\theta_{GMST}$

$0^\circ$

$\gamma$

$GHA_{\gamma}$
Hour Angles vs Time?

• 24 hrs = 360 degrees
  – Sidereal time assumed
Hour Angles
What Time is it? (continued)

• Additional times
  – UT1 (Universal Time, sidereal time)
    • Solution from observations
    • Shows slowly decreasing Earth rotation rate
  – UTC is Coordinated Universal Time (solar time)
    • “Clock time”
    • Maintained within 0.9 s of UT1
      – Leap Seconds
    • UTC = UT1 + ΔUT1
      – ΔUT1
        » EOP Parameter that accounts for actual Earth rotation
        » Calculated by USNO/ IERS
Time Scales
Summary for time

- Time
  - Can be off by up to a second if no $\Delta UT1$
  - TT can be off a minute
    - Used for many calculations
- Impact
  - Seems small but …
    - Consider satellite traveling at 7 km/s
- Many conversions necessary
  - Satellite moves wrt sidereal time
  - Clocks record Solar time
Coordinate Systems

- Sun based
  - Heliocentric
  - Barycentric
- Earth Based
  - Geocentric (Inertial and fixed)
  - Topocentric (fixed)
- Satellite Orbit Based
  - Perficoal
  - Radial vs Normal
  - Equinoctial
- Satellite Based
  - Attitude
Heliocentric Coordinate System

- Summer solstice
  - 1st day of Summer
  - ~ Jun 21
- Vernal equinox
  - 1st day of spring
  - ~ Mar 21
- Perihelion
  - ~ 1 Jan
- Aphelion
  - ~ 1 Jul
- Autumnal Equinox
  - 1st day of Fall
  - ~ Sep 23
- Vernal Equinox
  - 1st day of winter
  - ~ Dec 21
Geocentric and Ecliptic Coordinates
Geocentric Coordinate System
Local Coordinate System
Orbit Based Systems - Perifocal

Perigee, closest point to Earth
Orbit Based Systems – Normal and Radial

\[ \hat{N}, \hat{I}, \hat{J}, \hat{K} \]  
\[ \hat{W}, \text{cross-track} \]  
\[ \hat{V}, \hat{R}, \text{radial} \]  
\[ \hat{S}, \text{along-track} \]  
\[ \hat{v}, \hat{T}, \text{in-track} \]
Orbit Based Systems - Equinoctial
Angular Measurements

• Latitude and longitude
  – Familiar

• Right Ascension-Declination
  – Optical measurements
Right Ascension - Declination
Motion of the Coordinate System

• Earth’s orbit is not exactly stable
  – Precession
    • Long period movement (~26000 years)
  – Nutation
    • Short period movement (~18.6 years)

• Fixed vs Inertial
  – Sidereal Time

• Polar Motion
  – Axis of rotation moves slightly over time
Precession and Nutation

Luni-solar precession effect

Nutation effect

Ecliptic plane

Planetary effect

Earth’s equator

Precession of Equinox

Earth’s orbit about Sun
Polar Motion

[Diagram showing polar motion with axes and labels such as CEP, IRP, \( I_{PEF} \), \( \hat{I}_{ITRF} \), and \( \lambda = 90^\circ W \)].
Celestial Reference Frame

- MHB2000 ($z_A, \theta_A, \zeta_A$)
- MHB2000 ($\chi_A, \omega_A, \psi_A, \epsilon_0$)

Terrestrial Reference Frame

- X, Y
- X, Y

Bias-Precession
Nutation

- [BPN]

Sidereal Rotation

- $R_3[\theta_{GAST \cdot 2000}]

Polar Motion

- $R_1[-\gamma_p] R_2[-x_p]$

2003 Procedures

Traditional
Canonical 4-term Rotation
Non Rotating Origin
Series
Traditional Interpolation

2003 Procedures

Traditional
Canonical 4-term Rotation
Non Rotating Origin
Series
Traditional Interpolation

www.centerforspace.com
Terrestrial Reference Frame

Terrestrial Reference Frame

Bias-Precession
Nutation

Sidereal Rotation

Polar Motion

MOD

CIRS

ERS

TIRS

ITRF

Traditional

Canonical 4-term Rotation

Fukushima – Williams

Non Rotating Origin

Series

Traditional Interpolation

X, Y, s (Series) or EO

Equation (xxx) = f (X, Y, s)

R3[θERA·2006]

R3[θERA]

R3[s']

R1[-yp] R2[-xp]

R1[-yp] R2[-xp] R3[s']

R1[θGAST·2006]

MHB2000 (Δε, Δψ, ε₀)
+ optional 2006 rate adjustments

P03 (zA, θA, ζA)

P03 (χA, ωA, ψA, ε₀)

P03 (εA, ψJ, φJ, γJ)

2006 Procedures

2006 Procedures
Earth’s shape

- **Oblate Spheroid**
  - An ellipsoidal approximation

- **Other terms**
  - **Geoids**
    - Gravity acts equally at all points on this surface
      - Plumb-bobs will hang perpendicular
  - **Geopotential**
    - Mathematical representation of the precise gravitational effect
Earth Surface

Deflection of the Vertical

Mean Ocean Surface

\( H_{MSL} \)

\( h_{ellp} \)

Actual surface

Ellipsoid

\( \perp \) to plumb-bob

\( \perp \) to ellipsoid

Mean Sea Level (geoid)
Earth Ellipsoid

- Convert geocentric ($\phi_{gc}$) and geodetic ($\phi_{gd}$) latitude
Chapter 1

- Fundamental Concepts
- Newton
- Kepler
- Perturbations
- Orbit Determination
- Applications
Newton’s Laws

• 1. Every body continues in its state of rest, or of uniform motion in a right [straight] line, unless it is compelled to change that state by forces impressed upon it.

• 2. The change of motion is proportional to the motive force impressed and is made in the direction of the right line in which that force is impressed.

• 3. To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts. (Newton [1687] 1962, 13)

– The third law in Newton’s own words:

If a horse draws a stone tied to a rope, the horse (if I may say so) will be equally drawn back towards the stone; for the distended rope, by the same endeavor to relax or unbend itself, will draw the horse as much towards the stone as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other. (Newton [1687] 1962, 14)
Gravitational Law

• Forms the basis of Two-body dynamics
  – G is constant of gravitation = $6.673 \times 10^{-20}$ km$^3$/kgs$^2$

\[
\vec{f}_{\text{gravity}} = -\frac{Gm_{\oplus}m_{\text{sat}}}{r^2} \frac{\vec{r}}{|\vec{r}|}
\]
Two-body Equation of Motion

- Simple form resulting from

\[
\dot{\mathbf{r}} = -\frac{G(m_\oplus + m_{sat})}{r^2} \frac{\mathbf{r}}{|\mathbf{r}|}
\]
Chapter 2

- Fundamental Concepts
- Newton
- Kepler
- Perturbations
- Orbit Determination
- Applications
Kepler’s Laws

• 1. The orbit of each planet is an ellipse with the Sun at one focus.
• 2. The line joining the planet to the Sun sweeps out equal areas in equal times.
• 3. The square of the period of a planet is proportional to the cube of its mean distance to the Sun.
Conic Sections

- All orbits follow
  - Circle
  - Ellipse
  - Parabola
  - Hyperbola
  - Rectilinear
Nomenclature

• Kepler’s Equation and Kepler’s Problem
  – Very different!
  – Kepler’s equation
    • Found during Kepler’s analysis of the orbit of Mars
  – Kepler’s problem
    • Generically used for propagating a satellite forward
      – Usually two-body dynamics
Kepler’s Equation

- Find Eccentric anomaly \((E)\)
  \[-E = 0^\circ \text{ at } \nu = 0^\circ, \ 180^\circ\]
Kepler’s Problem

• Find future position and velocity
  – Given starting state
  – Called propagation
Satellite State Representations

- Convey location of a satellite in space and time
- Types
  - Numerical
    - Position and velocity vectors
  - Analytical (Elements)
    - Classical (Keplerian, Osculating, two-body) \((a, e, i, \Omega, \omega, \nu)\)
    - Equinoctial \((a_f, a_g, L, n, \chi, \psi)\)
    - Flight \((\lambda, \phi_{gc}, \phi_{fpa}, \beta, r, \nu)\)
    - Spherical \((\alpha, \delta, \phi_{fpa}, \beta, r, \nu)\)
    - Canonical
      - Delaunay
      - Poincare
  - Mean elements (theory dependant)
    - Two-line element sets
      » AFSPC, SGP4 derived, ‘mean’ elements
    - ASAP
    - LOP
    - Other
  - Other
    - Semianalytical
      - Theory dependant
Classical Orbital Elements

Angular momentum, $h$

Equatorial Plane

Perigee, $e$

Line of nodes, $\hat{n}$
• Fundamental Concepts
• Newton
• Kepler
• Perturbations
• Orbit Determination
• Applications
Introduction

• Several forces affect satellite orbits
  – Gravitational
  – Atmospheric Drag
  – Third Body
    • Sun, Moon, planets
  – Solar Radiation Pressure
  – Tides
    • Solid Earth, Ocean, pole, etc.
  – Albedo
  – Thrusting
  – Other
Perturbations

<table>
<thead>
<tr>
<th>Time</th>
<th>Mean Change</th>
<th>Short-periodic plus long-periodic, and secular</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_3$</td>
<td>Mean Change</td>
<td></td>
</tr>
<tr>
<td>$t_4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$c$
Central Body Gravitational Forces

• Largest single contributor to the motion
  – It’s why satellites stay in orbit!

• Conservative force
  – Total kinetic and potential energy remains the same

\[
V = \frac{GM}{r} \left[ 1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left( \frac{a}{r} \right)^n P_{nm}(\sin \phi) \left( C_{nm} \cos m \lambda_E + S_{nm} \sin m \lambda_E \right) \right]
\]

\[
P_n(\sin \phi) = \frac{1}{2^n n!} \frac{d^n}{d^n(\sin \phi)} (\sin^2 \phi - 1)^n
\]

\[
P_{nm}(\sin \phi) = (\cos \phi)^m \frac{d^m}{d^m(\sin \phi)} P_n(\sin \phi)
\]

\[
P_{nm} = \left( \frac{(2n+1)k(n-m)!}{(n+m)!} \right)^{1/2} P_{nm}, \quad \text{and} \quad \left\{ \frac{C_{nm}}{S_{nm}} \right\} = \left[ \frac{(n+m)!}{(2n+1)k(n-m)!} \right]^{1/2} \left\{ \frac{C_{nm}}{S_{nm}} \right\}
\]

with \( k = 1 \) for \( m = 0 \), and \( k = 2 \) for \( m \neq 0 \).
Zonal Harmonics

2, 0 3, 0 4, 0

Top

Side

5, 0
Sectoral Harmonics

\[ \ell = 2 \quad \ell = 3 \quad \ell = 4 \]

Top

Side

\[ \ell = 5 \]
Tesseral Harmonics

Top

Side

6, 4

2, 1

3, 1

3, 2

4, 1
Nodal Regression
Nodal Regression

![Graph showing nodal regression with inclinations and angular rates.](image)
Apsidal Rotation
Apsidal Rotation
Gravitational Effects

• Long ago when computers were slow…
• Gravitational modeling
  – Often square gravity field truncations
    • Appears the zonals contribute more
  – Point to take away:
    • Use “complete” field
    • Any truncations should include additional, if not all, zonal harmonics
Gravitational Modeling

- Satellite JERS (21867)
  - Comparison to 12x12 field
  - Note the variability over time
    - 22x22 vs 18x18 and 70x22 vs 70x18
Atmospheric Drag

- Large force for near-Earth satellites
  - Very difficult to model

- Non-conservative force
  - Total kinetic and potential energy not constant
    - Heat, other losses through friction

\[
\ddot{a}_{\text{drag}} = -\frac{1}{2} \rho \frac{c_D A}{m} v_{\text{rel}}^2 \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|}
\]
Drag Effect on Orbits

Orbit tends to circularize
Available Data
Space Weather – Predictions

• Lots of Variability
  – Constant $F_{10.7}$
    • Not very accurate
    • Never use 0.0!
  – Schatten
    • Varies with each solar cycle
  – Polynomial Trend
  – Matches several solar cycles
    • $F_{10.7} = 145 + 75 \times \cos(0.001696 \times t) + 0.35 \times \sin(0.001696 \times t)$
    – $t$ is the number of days from Jan 1, 1981
Observed vs Adjusted Solar Flux

- Data errors
  - Some inconsistencies
    - 10-40 SFU
  - Which does the model require?
    - MSIS
      - Observed
    - Others
      - Adjusted

DRAO (obs) - DRAO (adj) data
DRAO (adj) - Lenhart (adj) data
Solar Flux Predictions – Long Term

- Data differences
  - One solar cycle
    - ~150 SFU
  - Almost equal to the solar min-max difference!
Solar Flux Predictions – Shorter Term

- Data differences
  - Min, Mid, and Max
  - 30-50 SFU
  - Note timing of Cycle is off
Solar Flux Predictions – Shorter Term

- Early, Mid, and Late
- Also 30-50 SFU differences
Solar Flux Predictions – Short Term

- NOAA Predictions
  - 27-day and 45-day ($F_{10.7}$ and $a_p$)
  - 3-day
    - 3-hourly $K_p$ values off significantly as well
Simulated Sensitivity Analysis

- JERS sample orbit
  - Different atmospheric models
    - Baseline
      - Numerical propagation
      - Jacchia-Roberts
      - 3-hourly geomagnetic
    - Relative comparison only

![Graph showing differences in time from Epoch]
Simulated Sensitivity Analysis

- JERS sample orbit
  - Different treatment of the data
  - Baseline
    - Numerical Propagation
    - Jacchia-Roberts
    - 3-hourly geomagnetic
NRLMSISE-00 Results – Short Term
NRLMSISE-00 Results – Long Term

• Observations:
  – Model specifies observed
    • Adjusted performed well
  – Centered 81-day best
  – 20:00 UTC best
  – Spline interpolation very good

• No single best answer
Jacchia-Roberts Results – Short Term

![Graph showing time vs. difference (m) for various parameters](image-url)

- Parameters: LAT17, LAS20, LAT20, LAD20, LAI20, LOS20, LOD20, LOT17, LOT20, COT17, COS20, COD20, COT20, COI20, ObsConAllAvg, L81ObsConAll, CAT17, CAT20, CATI20, CAT20, CAS20, COS20, COI20, LOT20, LOT17, LOS20, LOD20

*Time, min from Feb 20, 2008 00:00:00.000 UTC*
Jacchia-Roberts Results – Long Term

- Observations:
  - Adjusted performed well in all cases
  - Centered 81-day best
  - 20:00 UTC best
  - Daily geomagnetic very good, but all were close

- No single best answer
Atmospheric Drag Effects

• Atmospheric Drag
  – Large variations
    • Changing the atmospheric model
    • Changing how the input data is interpreted
      – $F_{10.7}$ at 2000 UTC
      – Last 81-day average $F_{10.7}$ vs. the central 81-day average
      – Using step functions for the atmospheric parameters vs interpolation
      – Many others (see AIAA and UC paper)
  – Point to take away:
    • 1-1000 km differences are possible
    • Unable to determine if from data interpretation or model differences
Third Body Forces

- Can affect GEO satellites strongly
- Conservative force (like gravity)

\[ \ddot{a}_{3\text{-body}} = -\frac{G(m_{\oplus} + m_{\text{sat}})r_{\oplus\text{sat}}}{r_{\oplus\text{sat}}^3} + Gm_3\left(\frac{\vec{r}_{\text{sat}3}}{r_{\text{sat}3}^3} - \frac{\vec{r}_{\oplus3}}{r_{\oplus3}^3}\right) \]
Solar Radiation Pressure

- Large effect for high altitude satellites (GPS, GEO, etc)
  - Non conservative force
- Shadowing by the Earth becomes very important
  - All satellite altitudes
- Solar Irradiance \( (p_{sr}) \) is difficult to measure accurately

\[
\vec{a}_{srp} = -p_{SR} \frac{c_R A_{Sun}}{m} \frac{\vec{r}_{Sat-Sun}}{\left| \vec{r}_{Sat-Sun} \right|}
\]
Solar Irradiance (W/m²)
Earth Shadow Geometry

PLANET CALVIN MOVES ACROSS THE SOLAR SYSTEM.

Nobody notices until his orbit takes him directly between the sun and earth.

Calvin causes a total solar eclipse! Earth is shrouded in darkness. How long will Calvin stay there?

Could you move, please? You're in my light.

HA HA HAAA!
Earth Shadow Geometry

- Sun
- Earth
- Penumbra
- Umbra
- Nautical 102°
- Civil 96°
- Twilight 90°
- Astronomical 108°
- Lunar Orbit
- GEO Orbit
- Earth
- Penumbra
- Umbra
Solar Radiation Pressure Sensitivity

Results

• Solar Radiation Pressure
  – Several variations shown
    • 26690 (GPS)
  – Notice max is ~100m
  – Definitions
    • Cylindrical
      – Defines shadow type
    • App to true
      – Acct for light travel from Sun to CB
    • True
      – Inst light from Sun
    • No Boundary
      – Change step size at penumbra/umbra
  – Point to take away
    • Relatively small effect
    • Some variations
Special Perturbations

- Numerically integrate the equations of motion
  - Time consuming, but accurate

\[ \ddot{a} = \frac{\mu r}{r^3} \dot{r} + \ddot{a}_{\text{non-spherical}} + \ddot{a}_{\text{drag}} + \ddot{a}_{3\text{-body}} + \ddot{a}_{\text{srp}} + \ddot{a}_{\text{tides}} + \ddot{a}_{\text{other}} \]
General Perturbations

• Truncate analytical expansions and solve directly
  – Large time steps
• Each approach is mathematically different
  – SGP4
Semianalytical

- Blend numerical and analytical
  - Analytically solve secular and long period components
  - Numerically integrate the small short period variations

![Graph showing mean change, short-periodic plus long-periodic, and secular components over time.]

\[ F \]
Force Model Sensitivity Results

• Force model contributions
  – Determine which forces contribute the largest effects
    • 12x12 gravity field is the baseline
  – Note
    • Gravity and Drag are largest contributors for LEO
    • 3rd body ~km effect for higher altitudes

– Point to take away:
  • Trying to get the last cm from solid earth tides no good unless all other forces are at least that precise
Force Model Contributions

- Low Earth Orbit

ISS

25544

JERS

21867
Force Model Contributions

- Low Earth Orbit

Starlette

Vanguard II
Force Model Contributions

- Low to Mid Earth Orbit

Topex

GPS

22076

26690

Difference (m)

Time, min from Epoch

0 1440 2880 4320 5760

0 10000.0 100000.0 1000000.0

0 100.0 1000.0 10000.0 100000.0

0 10.0 100.0 1000.0 10000.0

0 1.0 10.0 100.0 1000.0

0 0.1 1.0 10.0 100.0

Time, min from Epoch

0 1440 2880 4320 5760

0 10000.0 100000.0 1000000.0

0 100.0 1000.0 10000.0 100000.0

0 10.0 100.0 1000.0 10000.0

0 1.0 10.0 100.0 1000.0

0 0.1 1.0 10.0 100.0
Force Model Contributions

- Mid Earth Orbit, eccentric
  SL 12 RB
  Molnyia
Force Model Contributions

• Low Earth and Geosynchronous Orbit

ISS (for comparison)  
Galaxy 11
Chapter 10

• Fundamental Concepts
• Newton
• Kepler
• Perturbations
• Orbit Determination
• Applications
Terms

• Orbit Determination
  – Process of determining an orbit from observations
  – Also called Estimation

• Filtering
  – Determining the current state after each observation

• Smoothing
  – Improve previous state solutions using future data
  – Runs backwards
Terms

• Deterministic
  – Dynamics are known and can be calculated
  – Propagation
    • Assuming a specific set of force models

• Stochastic
  – Uses observations to correct for unknown or mis-modeled dynamics
Terms

• Least Squares
  – Minimizes the sum-square of the residuals
  – Depends on a fit span
    • Length of time to process a batch of observations
  – Often called Batch Least Squares (BLS)
Linear Least Squares Example

• Assume a mathematical model of motion
  \[ y = \alpha + \beta x \]

• Residuals defined as
  \[ r_i = y_{o_i} - y_{c_i} = y_{o_i} - (\alpha + \beta x_{o_i}) \]

• Cost function (Jacobian)
  \[ J = \sum_{i=1}^{N} \bar{r}_i^2 = f(\alpha, \beta) = \sum_{i=1}^{N} (y_{o_i} - (\alpha + \beta x_{o_i}))^2 \]

• Minimization of residuals
Linear Least Squares Example

- **Matrix development**

\[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
x_{o_1} & x_{o_2} & \ldots & x_{o_N}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
x_{o_1}
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
x_{o_1} & x_{o_2} & \ldots & x_{o_N}
\end{bmatrix}
\begin{bmatrix}
y_{o_1} \\
y_{o_2} \\
\vdots \\
y_{o_N}
\end{bmatrix}
\]

\[
A^T A X = A^T b
\]

- **Normal Equation**

\[
X = (A^T A)^{-1} A^T b
\]
Statistical Concepts

- Dimensions and probability

<table>
<thead>
<tr>
<th>Dimension</th>
<th>$z = 1\sigma$</th>
<th>$2\sigma$</th>
<th>$3\sigma$</th>
<th>$4\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>68.27</td>
<td>95.45</td>
<td>99.73</td>
<td>99.99</td>
</tr>
<tr>
<td>2</td>
<td>39.35</td>
<td>86.47</td>
<td>98.89</td>
<td>99.96</td>
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<tr>
<td>3</td>
<td>19.87</td>
<td>73.85</td>
<td>97.07</td>
<td>99.89</td>
</tr>
</tbody>
</table>

$$erf\left(\frac{z}{\sqrt{2}}\right)$$

$$1 - \exp\left(-\frac{z^2}{2}\right)$$

$$erf\left(\frac{z}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}z \exp\left(-\frac{z^2}{2}\right)$$
Covariance Matrix

- Measure of uncertainty
- Grows with the satellite state propagation

\[ P = (A^T W A)^{-1} \]
- \( W \) is weighting or sensor accuracies
- \( A \) is partial derivative matrix

- Correlation Coefficients
  - Off diagonal terms

- Eigenvalues
  - Indicates each axis of the ellipsoid
LS Applied to Satellites: Overview

**Initial Orbit Determination**
- How good?
  - Radius of Curvature
- What state representation?
  - Equinoctial, Keplerian, other

**Obtain Good Initial State Estimate, X**
- loop through observations

**Orbit Determination**
- How to solve for Jacobian?
  - Analytical, finite differencing
- Least Squares Solution method?
  - Classical, Single Value Decomposition

**Propagate X to observation times**
- Form Residuals
- Solve Jacobian
- Solve Least Squares
- Converged?
LS Algorithm: Matrix Inverse Approach

- **FOR i = 1 to the number of observations (N)**
  - Propagate (SGP4, HPOP) nominal state to time of observation (TEME, ICRF)
  - Find the slant range vector, sensor to the propagated state in the topocentric (SEZ) coordinate system
  - Determine nominal observations from the SEZ vector
  - Find the \( b \) matrix as observed – nominal observations
  - Form the \( A \) matrix
    - Finite (or central) differences
    - Analytical partials
      - \( H \), Partials depending on observation type
      - \( \Phi \), Partials for state transition matrix.
  - Accumulate \( A^TWA \) and \( A^T Wb \)
- **END FOR**
- Find \( P = (A^TWA)^{-1} \) using Gauss-Jordan elimination (LU decomposition and back-substitution)
- Solve \( \delta x = PA^T Wb \)
- Check RMS for convergence
- Update state \( X = X + \delta x \)
- Repeat if not converged using updated state
Sequential Batch Least Squares

- Process additional observations
  - Use previous results
- Bayes Theorem

- Normal Equation
  - This is for “k” previously determined obs
  - “k + n” new obs

\[
\delta x(0 \mid k + n) = (A_{\text{new}}^T W_{\text{new}} A_{\text{new}} + \hat{P}_k^{-1})^{-1} (A_{\text{new}}^T W_{\text{new}} \tilde{b}_{\text{new}} + A_k^T W_k \tilde{b}_k)
\]

\[
\hat{P}_{k+n} = \hat{P}(0 \mid k + n) = (A_{\text{new}}^T W_{\text{new}} A_{\text{new}} + \hat{P}_k^{-1})^{-1}
\]
Extended Kalman Filter
Extended Kalman Filter

at each obs time $H_{k+1} = \frac{\partial z}{\partial \hat{X}_{k+1}}$

Prediction

$\hat{X}(t_{k+1}, t_k) = \int_{t_k}^{t_{k+1}} \hat{\dot{X}}_t dt + \hat{X}_t$

$F = \frac{\partial \hat{\dot{X}}_{k+1}}{\partial \hat{X}_{k+1}}$

$\dot{\Phi}(t_{k+1}, t_k) = F(t)\Phi(t_{k+1}, t_k)$

$\delta \bar{x}_{k+1} = 0$

$\bar{P}_{k+1} = \Phi \bar{P}_k \Phi^T + Q$

Update

$\tilde{b}_{k+1} = z - H_{k+1} \bar{X}_{k+1}$

$K_{k+1} = \bar{P}_{k+1} H_{k+1}^T [H_{k+1} \bar{P}_{k+1} H_{k+1}^T + R]^{-1}$

$\delta \hat{x}_{k+1} = \delta \bar{x}_{k+1} + K_{k+1} \tilde{b}_{k+1}$

$\hat{P}_{k+1} = \bar{P}_{k+1} - K_{k+1} H_{k+1} \bar{P}_{k+1}$

$\hat{X}_{k+1} = \bar{X}_{k+1} + \delta \hat{x}_{k+1}$
Averaging and Fit Spans

- Obs are taken periodically
- Updates often occur at regular intervals
- Least Squares approaches “average” data collected for a “batch” of time – the Fit Span
• Fundamental Concepts
• Newton
• Kepler
• Perturbations
• Orbit Determination
• Applications
Applications

• How do we put all this together and accomplish our original goal?
  – Many analyses possible
    • Prediction
      – Satellite look angles (Our original question)

• Behind the scenes
  – Time of observations
  – Coordinate systems throughout
  – Orbit determination of observations to obtain a state vector
  – Propagation to form an ephemeris
  – Calculations for Sun and Satellite to determine visibility
  – ...
  – And several other smaller details!
Satellite Orbital Characteristics
Predicting Satellite Look Angles

Terminator

Site horizon
Earth shadow

Dist

\( r_\Theta \)

\( \omega_\Theta \)

\( r_{\text{site}} \)

\( \hat{\rho}_c \)

\( \hat{\rho}_b \)

\( c - \text{Night} \)

\( b - \text{Sunlight} \)

\( a - \text{Below horizon} \)
Rise Set Characteristics
Finding the Site Information (1)

• Approximate formulation
  – Non-rigorous ECEF
  – Don’t account for sidereal/solar time differences

\[
\begin{align*}
\dot{\rho}_{\text{ECI}} &= [\text{ROT3}(-\theta_{\text{LST}})][\text{ROT2}(-90^\circ - \phi_{gd})]\dot{\rho}_{\text{SEZ}} \\
\ddot{r}_{\text{ECI}} &= [\text{ROT3}(-\theta_{\text{LST}})][\text{ROT2}(-90^\circ - \phi_{gd})]\ddot{r}_{\text{SEZ}} \\
\ddot{r}_{\text{ECI}} &= \rho_{\text{ECI}} + \rho_{\text{SiteECI}} \\
\ddot{v}_{\text{ECI}} &= \dot{\rho}_{\text{ECI}} + \Omega_\oplus \times \ddot{r}_{\text{ECI}}
\end{align*}
\]
Finding the Site Information (2)

- Rigorous formulation (STK approach)
  - Precise ECEF
  - Account for sidereal/solar time differences

\[ \dot{\rho}_{ECEF} = [\text{ROT}3(-\phi)] [\text{ROT}2(-90^\circ - \phi_{gd})] \dot{\rho}_{SEZ} \]
\[ \ddot{\rho}_{ECEF} = [\text{ROT}3(-\phi)] [\text{ROT}2(-90^\circ - \phi_{gd})] \ddot{\rho}_{SEZ} \]

\[ \ddot{r}_{ECEF} = \ddot{\rho}_{ECEF} + \ddot{r}_{SiteECEF} \]
\[ \ddot{v}_{ECEF} = \ddot{\rho}_{ECEF} \]

(yr, mon, day, UTC, ΔUT1, ΔAT) \Rightarrow (UT1, TAI, TT, T_{UT1}, T_{TT})

[PREC] = \text{ROT3}(-z) \text{ROT2}(\Theta) \text{ROT3}(-\zeta)

[NUT] = \text{ROT1}(\epsilon) \text{ROT3}(\Delta \Psi) \text{ROT1}(\Phi)

[ST] = \text{ROT3}(\theta_{AST})

[PM] = \text{ROT2}(-x_p) \text{ROT1}(-y_p)

\[ \ddot{r}_{ECI} = [\text{PREC}]^T [\text{NUT}]^T [\text{ST}]^T [\text{PM}]^T \ddot{r}_{ECEF} \]
\[ \ddot{v}_{ECI} = [\text{PREC}]^T [\text{NUT}]^T [\text{ST}]^T \left\{ [\text{PM}]^T \ddot{v}_{ECEF} + \ddot{\omega}_\oplus \times \ddot{r}_{PEF} \right\} \]
Results

• Rigorous approach
  – Position (ECI)
    • -5505.504883 km
    • 56.449170
    • 3821.871726

• Simplified approach
  – Position (~ECI)
    • -5503.79562 km
    • 62.28191
    • 3824.24480

• Difference
  – 6.52 km

• Perhaps this is acceptable?
Impact

• Applying textbook solutions to real-world problems will give the wrong answers
  – Assumptions add up
  – Examples:
    • Communicating with a satellite using Laser comm
      – At orbital velocity, 2 sec is nearly 14 km
        » Will your signal be able to locate and receive?
    • Will you pass System Acceptance Testing?
A word of Caution …

- Fundamentals vs Applications
  - Undergraduate vs Graduate
  - Classroom vs Operational
  - Attention to detail important
  - Nomenclature is important
Resources

• Book
  – Microcosm
    • Pam is here!

  – TLE data
  – EOP and Space Weather Data
  – Code
    • SGP4
    • Other
  – Errata
    • Not all updated but most are
  – Solutions
    • Not complete – ask 😊
Questions??

I was wrong, Cliff, you DO have to be a rocket scientist to figure this out!